Computational Complexity of k-stratified Graph Construction

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Abstract

A graph $G = (V, E)$ is a k-stratified graph if V is partitioned into V_1, V_2...V_k classes (strata). Let G_i be the subgraph of G induced by V_i $(1 \leq i \leq k)$ and G_{ij} be the bipartite subgraph with V_i, V_j its two partites and $E_{ij} = \{(u, v) \in E | u \in V_i, v \in V_j\}$. If d is the degree of a vertex $v \in V$, then k-stratification of G decomposes d to a k-vector of integers (d_1, d_2,...,d_k), where d_i $(1 \leq i \leq k)$ indicates the number of vertices in V_i incident to v. Note that $d = \sum_{i=1}^{k} d_i$. If a multiset of n k-vectors is given does a k-stratified graph exist which defines these vectors? The computation complexities of this problem and a number of its subproblems are investigated. It is proven that the above problem is NP-Complete. It remains NP-Complete even if the cardinalities of V_1, V_2,...,V_k are known. The problem becomes polynomial if the identities of vertices in each V_i $(1 \leq i \leq k)$ are known.

Key words: k-stratified graphs, NP-complete, complexity, integer vectors.

1 Introduction

Graphs are frequently used to model different computer structures. The Very Large Scale Integrated Circuit (VLSI) is a complex computer structure modeled by graphs. VLSI chips contain millions of transistors and connecting wires. The design of a circuit on a VLSI chip involves a step that maps its specifications to a geometric structure. This step is called the circuit layout and the mapping is called the physical design [13]. Problems encountered in the physical design are often represented as graph theoretical problems. Such problems include minimization problems [6] [11], routing problems [13] [14] and stratification problems [10] [12]. The advancement in technology has allowed the use of more than one layer (stratum) for routing interconnections [9]. The routing algorithms for multilayer routings can be conveniently be designed using k-stratified graphs as graph representations [10].

Complexity answers to questions related to design routings can be greatly facilitated by the representation of a graph as layered (stratified graphs) [1] [2]. A special class of stratified graphs called uniform stratified graphs has been investigated in [8]. The uniform stratified graphs have the property that the degrees of each vertex relatively to each of the strata are equal [2]. Again, for this class of stratified graphs some complexity answers are greatly facilitated.

This paper deals with the intractability of some of the problems involved in the design modeled by k-stratified graphs. Different restrictions are introduced and their effect in the intractability of the resulting sub problems are investigated. A spectrum of sub problems are considered. The spectrum of these sub problems moves progressively from intractable to polynomial problems.

2 Preliminaries

In this section we define some of basic terms that are used to describe the problems dealt in this paper.

Definition 1. *A graph $G = (V, E)$ is a k-stratified graph if V is partitioned into $V_1, V_2 ... V_k$ classes (strata). Let G_i be the subgraph of G induced by V_i $(1 \leq i \leq k)$ and G_{ij} be the bipartite subgraph with V_i, V_j its two partities and $E_{ij} = \{(u, v) \in E | u \in V_i, v \in V_j\}$.*

For $k = 2$ the graph is called a 2-stratified graph and V is partitioned into two classes V_1 and V_2. G_1 and G_2 are subgraphs induced by V_1 and V_2. G_{12} is the bipartite subgraph with V_1 and V_2 its two partites. Figure 1 shows an example of a 2-stratified graph with $V_1 = \{a, b, c\}$ and $V_2 = \{d, e, f, g\}$.

Definition 2. *The degree of a vertex is defined as the number of edges incident on the vertex.*

Definition 3. *If d is the degree of a vertex $v \in V$, then k-stratification of G decomposes d to a k-vector of integers $(d_1, d_2, ..., d_k)$, where d_i $(1 \leq i \leq k)$*

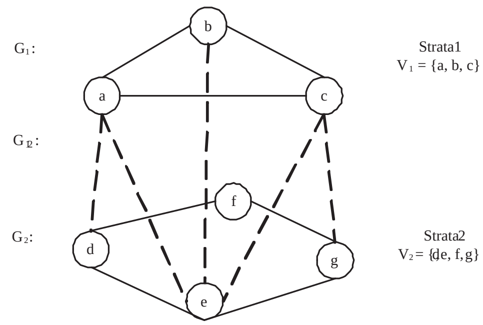

Figure 1: An example of a 2-stratified graph $G(V, E)$. The dotted lines represent the edges E_{12}

indicates the number of vertices in V_i incident to v. The summation of d_i's gives the degree d.

From figure 1, the 2-vector for vertex a is (2,1). It can be seen that 2 vertices in V_1 and 1 vertex in V_2 are incident to a. Therfore, by definition, the 2-vector for a is (2,1).

Definition 4. *For a k-stratified graph $G = (V, E)$ with n vertices, a multiset is defined which includes all n k-vectors for G.*

The multiset is denoted by the symbol D. The multiset for the 2-stratified graph shown in Figure 1 is, $D = \{(2, 1), (2, 2), (2, 2), (1, 2), (0, 2), (3, 2), (1, 2)\}$ Note that a multiset could contain an element many times.

Definition 5. *A uniform graph is a k-stratified graph in which a vertex has the same degree for each stratum.*

This implies that for a uniform graph, the integer components for each k-vector of the multiset are identical. The graph in figure 2 is a 2-stratified uniform graph with $D = \{(2, 2), (2, 2), (2, 2), (2, 2), (1, 1), (2, 2), (1, 1)\}$.

Definition 6. *A regular graph is one in which, all of its vertices have the same degree.*

A uniform-regular graph will then have the properties of both uniform and regular graphs. This implies that for a uniform-regular graph, the integer terms within each k-vector are identical and the k-vectors for all the vertices are also identical to each other. The graph in Figure 3 is 2-stratified uniform-regular graph with $D = \{(2, 2), (2, 2), (2, 2), (2, 2), (2, 2), (2, 2)\}$.

Definition 7. *The partition problem states that given a finite set A and a size $s(a) \in Z^+$, $\forall a \in A$, is there a subset $A' \in A$ such that $\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)$. Partition problem has been proven to be NP-Complete [4]. The partition problem is denoted as PART.*

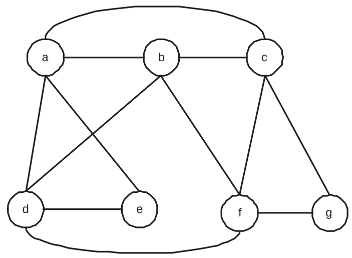

Figure 2: An example of a 2-stratified uniform graph

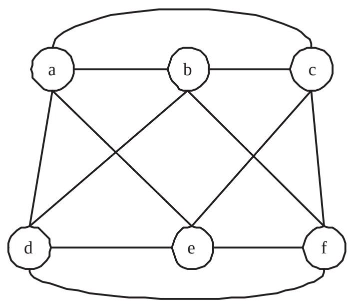

Figure 3: A 2-stratified uniform-regular graph

3 k-stratification Problems

The general k-stratified graph construction problem is defined as follows: Given a multiset D of n k-vectors is there a k-stratified graph G with its multiset of vectors identical to D? In this paper we also discuss some restrictions to the general problem. The problem is analyzed when the multiset is restricted to a uniform graph and a uniform-regular graph. Another restriction provides the cardinality of the vertices in each stratum and analyzes the resulting complexity of the problem. Finally, the fourth restriction provides the identities of the vertices that belong to each stratum. The formal definitions of the above problems closely follow the notation found in [4].

Π_1. *Given a multiset D of n k-vectors of integers and $k \geq 2$, is there a k-stratified graph $G = (V, E)$ with its multiset of vectors identical to D?*

Restricting k to be equal to 2, the minimum of its possible values, the sub problem Π_2 of Π_1 is defined as follows:

Π_2. *Given a multiset D of n 2-vectors of integers, is there a 2-stratified graph $G = (V, E)$ with its multiset of vectors identical to D?*

If it is known that n_1 of the n vectors belong to the first stratum and therefore $n_2 = n - n_1$ belong to the second stratum then a subproblem Π_3 of Π_2 is defined as follows:

Π_3. *Given a multiset D of n 2-vectors of integers, and an integer $n_1 < n$, is there a 2-stratified graph $G = (V, E)$ with $|V_1| = n_1$, $|V_2| = n_2 = n - n_1$ such that its multiset of vectors is identical to D?*

Given $D = \{(2,1), (2,2), (2,2), (1,2), (0,2), (3,2), (1,2)\}$ and $n_1 = 3$, is there a 2-stratified graph with $|V_1| = 3$ and $|V_2| = 4$? Though the cardinality of the vertices in each strata is given, the identity of the vertices which vertices belong to V_1 and V_2 need to be determined.

If it is also known which n, vectors belong to one stratum, instead of how many, then a subproblem Π_4 of Π_3 is defined as follows:

Π_4. *Given a multiset D of n 2-vectors of integers and the identities of $n_1 < n$ of the vectors which belong to one stratum, is there a 2-stratified graph $G = (V, E)$ with V_1 containing the identified vectors, V_2 containing the rest $n_2 = n - n_1$ and such that the multiset of G is identical to D?*

Given $D = \{v_1 = (2,1),\ v_2 = (2,2),\ v_3 = (2,2),\ v_4 = (1,2),\ v_5 = (0,2),\ v_6 = (3,2),\ v_7 = (1,2)\}$, $n_1 = 3$, $V_1 = \{v_1, v_2, v_3\}$ and V_2 includes the remaining vertices, is there a 2-stratified graph with V_1 and V_2 as defined? In this sub problem, the identity of the vertices which belong to each stratum is also known.

Restrictions could also be imposed on other aspects of the vector set D. If all the integer components of a vector are equal, then the subproblem Π_5 of Π_1 is defined as follows:

Π_5. *Given a multiset D of n k-vectors of integers and $k \geq 2$ and $\forall d_i \in D$; $d_{i_1} = d_{i_2} = \ldots = d_{i_k}$, is there a k-stratified graph $G = (V, E)$ with its multiset of vectors identical to D? Note that Π_5 restricts the graph $G = (V, E)$ to be a uniform k-stratified graph.*

Given $D = \{v_1 = (2,2,2),\ v_2 = (1,1,1),\ v_3 = (1,1,1),\ v_4 = (1,1,1),\ v_5 = (1,1,1),\ v_6 = (1,1,1),\ v_7 = (1,1,1),\ v_8 = (1,1,1),\ v_9 = (2,2,2),\ v_{10} = (1,1,1)\}$, is there a 3-stratified graph with its multiset of vectors identical to D? For this instance, there a exists a 3-stratification with

$V_1 = \{v_1, v_2, v_4\}$, $V_2 = \{v_3, v_5, v_8, v_6\}$ and $V_3 = \{v_7, v_9, v_{10}\}$. The resulting graph is a uniform 3-stratified graph as shown in Figure 4.

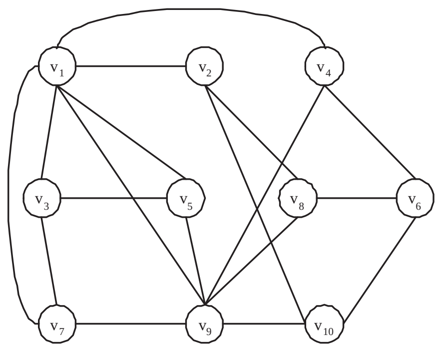

Figure 4: A uniform 3-stratified graph for the given D

When $k = 2$, Π_5 is restricted to a sub problem Π_6.

Π_6. *Problem* Π_5 *with* $k = 2$.

If the equality of the vector components also extends across all vectors then a sub problem Π_7 of Π_5 is defined as follows:

Π_7. *Given a multiset D of n k-vectors of integers with $k \geq 2$, $\forall d_i \in D$; $d_{i_1} = d_{i_2} = \ldots = d_{i_k}$ and $d_j = d_i$, $\forall d_i, d_j \in D$, is there a k-stratified graph $G = (V, E)$ with its multiset of vectors identical to D? Note that* Π_7 *restricts the graph to be a uniform and regular.*

Given $D = \{v_1 = (2, 2, 2),\ v_2 = (2, 2, 2),\ v_3 = (2, 2, 2),\ v_4 = (2, 2, 2),\ v_5 = (2, 2, 2),\ v_6(2, 2, 2),\ v_7 = (2, 2, 2),\ v_8 = (2, 2, 2),\ v_9 = 2, 2, 2)\}$, is there a 3-stratified graph with its multiset of vectors identical to D? For this instance, there a exists a 3-stratification with 3 vertices(any 3) in each stratum. The resulting graph is a uniform-regular 3-stratified graph as shown in Figure 5.

Π_8. *Problem* Π_8 *with* $k = 2$.

If $\Pi \to \Pi'$ represents the relationship, Π' is a sub problem of Π, then Figure 4 illustrates the relationships of all the problems. Note, that the relationship is transitive.

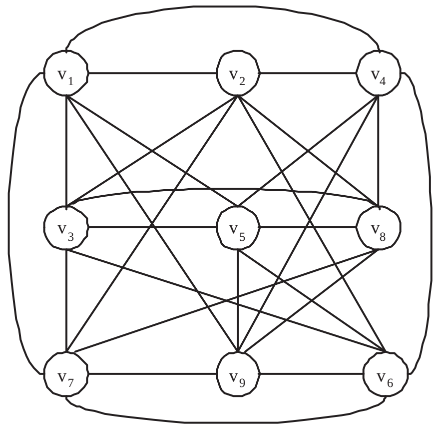

Figure 5: A uniform-regular 3-stratified graph for the given D

From the complexity theory it is known that if a problem Π_i is proven to be NP-Complete then all its predecessors are also NP-Complete. Also, if a problem Π_j is proven to be polynomial then all its subsequent problems are also polynomial. The intuition behind the above statements is that if a problem is intractable then any predeccesor(more difficult) problem is also intractable and if a problem is polynomial then any successor(easier) problem is also polynomial.

4 Problem Complexities

To prove that a problem Π is NP-Complete the following two steps are sufficient:

1. A guess which is a possible solution for an instance of Π must be polynomially verifiable.

2. A known NP-Complete problem Π' polynomially transforms to Π i.e. a "Yes" instance of Π' can be transformed on polynomial time into a "Yes" instance of Π and vice versa

The above transformation is called polynomial reduction of Π' to Π and is denoted as $\Pi' \, \alpha \, \Pi$.

To prove that a problem Π is polynomial it suffices to design a polynomial algorithm for Π.

Theorem 1. *Problem* Π_4 *is Polynomial.*

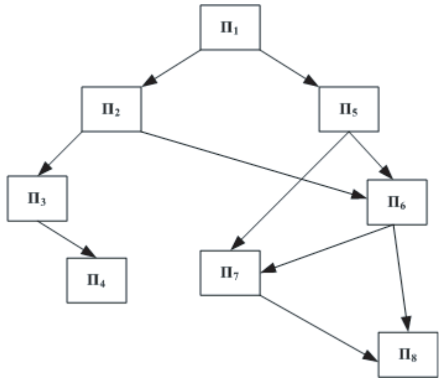

Figure 6: Relationship of the defined problems

Proof. It suffices to design a polynomial algorithm which answers the question of the problem.

Let $D = \{(d_{11}, d_{12}), (d_{21}, d_{22}), \ldots, (d_{n1}, d_{n2})\}$ with $n_1 < n$ vectors on stratum 1 and $n_2 = n - n_1$ on stratum 2. The following algorithm solves the problem.

Algorithm:

Step 1. Consider the first components of the n_1 vertices. Check if the resulting multiset is graphical. If it is graphical go to step 2 else answer "No" and stop, i.e. there is no 2-stratified graph according to D.

Step 2. Consider the second components of the n_2 vertices. Check if the resulting multiset is graphical. If it is graphical go to step 3 else answer "No" and stop.

Step 3. Consider the multiset that contains the second components of the n_1 vertices and the first components of the n_2 vertices. Check if the resulting multiset is graphical then the answer is "Yes" i.e. there is a 2-stratified graph according to D, else answer "No" and stop.

Each step of the algorithm can be carried out in polynomial time (graphicability is known to be polynomial). Therefore the algorithm is polynomial.

Step 1 constructs a graph G_1 if there exists one. Step 2 constructs a graph G_2 if there exists one. Step 3 constructs a bipartite graph G_{12} if there exists one.

The G_{12} has the n_1 vertices on one partite and the n_2 vertices on the other, Therefore, G_{12} connects G_1 and G_2 to form a 2-stratified graph with G_1 its first stratum and G_2 its second. Thus, the answer to Π_4 can be obtained in polynomial time. □

A positive example for the Theorem 1:

$D = \{v_1 = (2,2),\ v_2 = (2,1),\ v_3 = (4,1),\ v_4 = (2,1),\ v_5 = (2,2),\ v_6 = (2,1)\}$ with v_1, v_2, v_4, v_5 on stratum 1 and v_3, v_6 on stratum 2.

Step 1: $\{2,2,2,2\}$ is graphical.

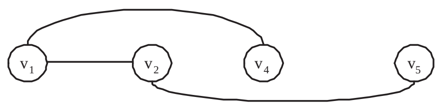

Figure 7: Construction of G_1

Step 2: $\{1,1\}$ is graphical.

Figure 8: Construction of G_2

Step 3: $\{1,2,1,2,4,2\}$ is graphical.

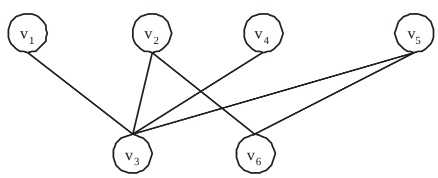

Figure 9: Construction of G_{12}

Therefore, the 2-stratified graph G according to D is as shown in Figure 10.

A negative example for Theorem 1:

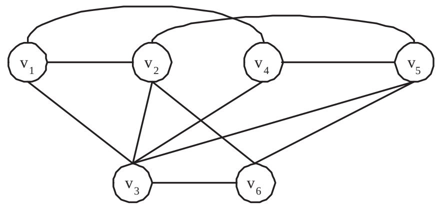

Figure 10: Construction of G

$D = \{v_1 = (4,3), v_2 = (1,1), v_3 = (2,2), v_4 = (3,3)\}$ with v_1, v_3, v_4 on stratum 1 and v_2 on stratum 2.

Step 1: $\{4,2,3\}$ is not graphical and thus there is no 2-stratified graph according to D.

Corollary 1. *The following problem* Π *is polynomial. Given a mulitset* D *of* n k*-vectors of integers,* $k > 2$ *and the identities of* $n_1, n_2, \ldots, n_k$ *of these vectors which belong to* $v_1, v_2, \ldots, v_k$ *respectively with* $n_1 + n_2 + \ldots + n_k = n$*, is there a* k*-stratified graph* $G = (V, E)$ *with strata* $v_1, v_2, \ldots, v_k$ *and such that the multiset of* G *is identical to* D*?*

Proof. Graphs $G_1, G_2, \ldots, G_k$ can be constructed polynomially if they exist. The construction is similar to the one described in the proof of Theorem 1. Also $\frac{k(k-1)}{2}$ bipartite graphs G_{ij}, where $1 \leq i \leq j \leq k$ can be constructed, if they exist as in Theorem 1.

Therefore the question of Π can be answered in polynomial time.

□

Theorem 2. *Problem* Π_6 *is NP-Complete.*

Proof. First it has to be proven that $\Pi \in NP$ i.e. guessing a possible solution from an instance of Π_6, it can be verified in polynomial time if the answer for this guess is "Yes" or "No". A guess will consist of a subset of vertices from D which belong to the first stratum ans thus the rest of them to the second stratum. The verification, if there is a graph G according to the guess with a multiset of n 2-vectors identical to D, can be done in polynomial time according to Theorem 1.

The second part of the proof consists of a polynomial reduction of the NP-Complete problem $PART$ to Π_6 i.e. $PART \, \alpha \, \Pi_6$.

Let A and $s : A \rightarrow Z^+$ be a generic instance of the PART. A specific instance for example is $A = \{a, b, c, d, e, f\}$ with $s(a) = 4$, $s(b) = 2$, $s(c) = 2$,

$s(d) = 1$, $s(e) = 1$ and $s(f) = 2$.

The polynomial reduction $PART \, \alpha \, \Pi_6$ involves the following two steps:

1. Construction in polynomial time of a uniform mulitset D of pairs of integers from an instance A of $PART$.

2. Proving that D is the multiset of a uniform 2-stratified graph if and only if A has a partition.

Step 1: Construction of D:

For every $x \in A$ with size $s(x)$ construct $D_x = \{(s(x), s(x))$, $(1,1)$, $(1,1)$, ...,$(1,1)$ $s(x)$ times, $(2,2)$, $(2,2)$, ...,$(2,2)$ $s(x)$ times$\}$. The size of D_x is $|D_x| = 2s(x) + 1$

The multiset D will be the union of all mulitsets D_x, $\forall x \in A$ i.e.,

$$D = \bigcup_{x \in A} D_x$$

The size of D is $|D| = \sum_{x \in A} |D_x| = 2 \sum_{x \in A} s(x) + |A|$. Thus, the size of D and therefore its construction is polynomial on the input length (input A, s).

For the instance of $PART$ given above the multiset D will be, $D =$ {(4,4), (1,1), (1,1) (1,1), (1,1), (2,2), (2,2), (2,2), (2,2), (2,2), (1,1), (1,1), (2,2),(2,2), (2,2), (2,2), (2,2), (1,1), (1,1),(2,2), (2,2), (1,1), (1,1), (2,2), (1,1), (1,1), (2,2), (2,2), (1,1), (1,1)}.

Step 2: Proof for the if and only if:

Let A have a partition, then D will also be subject to a corresponding partition by the construction of D. For the above example $A' = \{a, b\}$ and $A - A' = \{c, d, e, f\}$ define a partition on A because $s(A') = s(A - A')$. Also $D' =$ {(4,4), (1,1), (1,1) (1,1), (1,1), (2,2), (2,2), (2,2), (2,2), (2,2), (1,1), (1,1), (2,2),(2,2)} and $D - D' =$ {(2,2), (1,1), (1,1),(2,2), (2,2), (1,1), (1,1), (2,2), (1,1), (1,1), (2,2), (2,2), (1,1), (1,1), (2,2),(2,2)} is a partition of D relatively to their first or second vector components.

A' and $A - A'$ of the partition of A define the first and second strata G_1 and G_2 of a graph G. The addition of an appropriate number of 1's and 2's in the construction of D guarantees the construction of G_1 and G_2. Note that the added 2-vertices (vertices with degree 2 in each strata) in each D' and $D - D'$ are connected to a ring which is always possible. For the example G_1 and G_2 are illustrated in Figure 11.

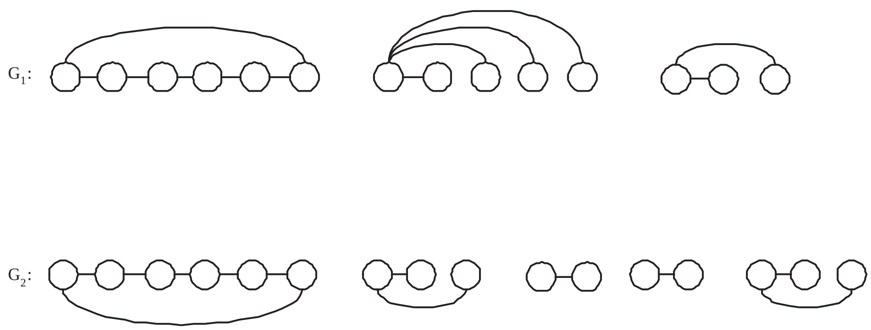

Figure 11: Graphs G_1, G_2 of G

Connect with edges the added 1-vertices of G_1 and the added 1-vertices of G_2. Also, connect the vertices in D' which correspond to elements of A with the added 2-vertices in the ring of G_2 and correspondingly the 2-vertices in the ring G_1. The constructed graph G according to the above is illustrated in Figure 12.

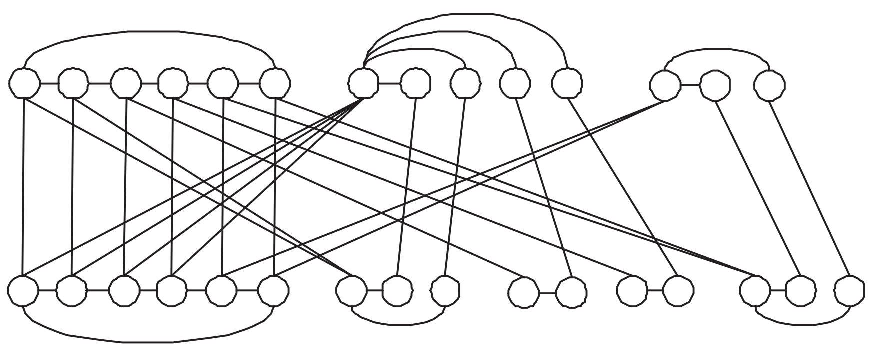

Figure 12: Graph G

The graph G is by construction a uniform 2-stratified graph which is constructed in polynomial time if A has a partition. The reverse is also true i.e. a uniform 2-stratified graph as above defines a partition between the vertices of the two strata. The removal of the extra 2's and 1's defines a partition on A. Therefore $PART \, \alpha \, \Pi_6$ and thus $\Pi_6 \in NP-Complete$. □

Corollary 2. *Problems* Π_1, Π_2, Π_5 *are NP-Complete.*

Proof. Π_6 which is proven to be NP-Complete is a sub problem to each one of them. Thus, Π_1, Π_2, Π_5 are NP-Complete. □

Theorem 3. *Problem* Π_8 *is Polynomial.*

Proof. The multiset of a uniform-regular graph has identical 2-vectors. For this multiset to reflect a 2-stratification, exactly half the number of vertices should belong to each stratum. Otherwise, the graph G_{12} will not be graphable. Since it is a uniform-regular graph, any half of the vertices can belong to each strata and this eliminates the exponential number of possibilities which exist otherwise.

Let $D = \{(3,3),(3,3),(3,3),(3,3),(3,3),(3,3),(3,3),(3,3)\}$. Since all the vectors are identical, any 4 vertices can belong to each one of the strata. Given the vertices that belong to each stratum, it can then be verified in polynomial time if the graphs G_1, G_2 and G_{12} are graphable (from Theorem 7). Therefore Π_8 is Polynomial. □

Corollary 3. *Problem* Π_7 *is polynomial.*

Proof. Extending the proof of Theorem 3 to k identical sets instead of 2, the answer to the question of Π_7 can be derived in polynomial time. □

Lemma 1. *If x is the degree of a vertex in each strata ($x = 3$ in above example) and there exists 2 strata, then the minimum uniform-regular graph should have exactly $2(x + 1)$ total vertices, with $(x + 1)$ vertices in each stratum.*

If x is the degree of a vertex in one strata, then it needs exaclty x other vertices in the same strata to satisfy the degree. Therefore $(x + 1)$ vertices is required in each stratum and $2(x + 1)$ vertices are required in total.

Lemma 2. *If there exists k-strata then the minimum uniform-regular graph should have at least $k(x+1)$ total vertices, with $(x+1)$ vertices in each strata.*

Lemma 3. *In a k-stratified minimum uniform-regular graph the sub graphs $G_1, G_2, \ldots, G_k$ are unique while the corresponding bipartite graphs are not uniquqe.*

By definition of a minimum uniform-regular graph, if the vertex degree in each stratum is x then there are exactly $(x + 1)$ vertices in each strata. Therefore to satisfy this degree requirement, every vertex will be incident with the remaining x vertices in the same strata. This implies that each strata will be a complete graph and therefore unique. The bipartite graph on the other hand requires a vertex in $strata_1$, to be incident with any x (among the $(x + 1)$) vertices in $strata_2$ and vice versa. Therefore the bipartite graphs are not unique.

Theorem 4. *Problem Π_3 is NP-Complete.*

Proof. Π_3 also belongs to NP. A polynomial Turing reduction technique can be used to prove that it is NP-Complete.

Assume Π_3 belongs to P (class Polynomial). Suppose that $S(D, n_1)$ is a subroutine for solving Π_3, with the parameters D - the multiset and n_1- the number of vertices that belong stratum 1. Π_2 can then be solved using the subroutine for Π_3. In other words, $\Pi_2 \; \alpha_T \; \Pi_3$.

Given D the multiset for Π_2, call the subroutine $S(D, 1)$. $S(D, 1)$ will answer if the given multiset represents a 2-stratified graph with 1 vertex in stratum 1 and remaining $(n-1)$ vertices in stratum 2. If $S(D, 1)$ returns the answer "yes", then it can be concluded that the multiset D represents a 2-stratified graph. If $S(D, 1)$ returns the answer "no" then the process is repeated with $S(D, 2)$. In this way, with at most $n/2$ calls to the subroutine, it can be determined if D represents a 2-stratified graph. (Note: $n/2$ calls will be needed only if the first $(n/2-1)$ calls to the subroutine answers "no"). The process is shown in Figure 13.

Based on the assumption, each call to the subroutine $S(D, n_1)$ requires polynomial time and at most $n/2$ calls to solve Π_2. Therefore, Π_2 can also be solved in polynomial time. However, it has been proved earlier that Π_2 is NP-Complete (Corollary 2) and cannot be solved in polynomial time unless P = NP. Therefore, the assumption Π_3 belongs to P is false.

This Turing reduction, $\Pi_2 \, \alpha_T \, \Pi_3$, shows that Π_3 is NP-HARD. Given Π_3 is NP and NP-HARD, it can be concluded that it is NP-Complete.

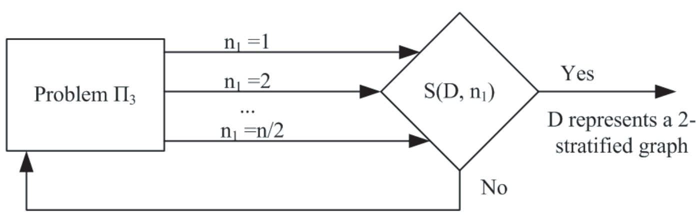

Figure 13: Turing Reduction for Π_2 using Π_3 as Oracle.

□

5 Conclusion

This paper discusses the complexity of k-stratification of a graph from a given multiset. A spectrum of sub problems were investigated. Formal techniques were used to prove that the complexity of these sub problems ranges from NP-Completeness to polynomial depending on the restriction.

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